Extremal Functions of Forbidden Matrices PRIMES Conference

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Definition

0,1...k-matrix: A matrix where all the entries are in $\{0,1...k\}.$

Example

$$\left(\begin{array}{rrrr}1 & 0 & 1\\ 1 & 0 & 1\\ 0 & 0 & 1\end{array}\right)$$

is a 0,1-matrix. It is also a $0,1\dots k\text{-matrix}$ for all k>1

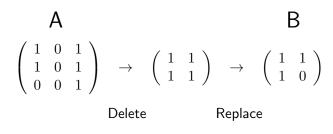
Containment and Avoidance for 0,1-Matrices



Definition

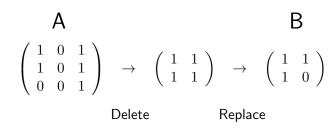
A contains B: We can delete rows and columns in A, and replace 1's with 0's to end up with B.

Containment and Avoidance for 0,1-Matrices



Therefore, A contains B.

Containment and Avoidance for 0,1-Matrices



Therefore, A contains B.

Definition

A avoids B: A does not contain B.

Definition

ex(P,n): If P is a 0,1-matrix, then this is the maximum number of 1's we can have in an $n \times n$ matrix that avoids P.

Example

$$P = \left(\begin{array}{rr} 1 & 1\\ 1 & 1 \end{array}\right)$$

 $\mathrm{ex}(P,3)=6,$ because any 3×3 matrix with over 6 ones contains P.

Theorem

$$Let P = \begin{pmatrix} 1 & 1 & 1 & . & . \\ 0 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . \\ . & . & . \end{pmatrix}$$

where P has r rows and c columns. Then, ex(P,n) = n(r+c) - (r-1)(c-1) Important: this is O(n)

Containment and Avoidance for 0, 1...k-Matrices

$$\begin{array}{ccc} A & & B \\ \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 2 \end{array} \end{pmatrix} \qquad \qquad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{array})$$

Definition

A contains B: We can delete rows and columns in A, and replace numbers with smaller numbers to end up with B.

Containment and Avoidance for 0, 1...k-Matrices

$$\begin{array}{ccc}
 A & B \\
 \begin{pmatrix}
 2 & 0 & 1 \\
 3 & 1 & 1 \\
 0 & 0 & 2
 \end{array}$$

$$\begin{array}{c}
 \begin{pmatrix}
 2 & 0 & 1 \\
 3 & 1 & 1 \\
 0 & 2
 \end{array}$$

Definition

A contains B: We can delete rows and columns in A, and replace numbers with smaller numbers to end up with B.

Definition

A avoids B: A does not contain B.

Definition

 $ex_k(P, n)$: The maximum sum of numbers an $n \times n$ 0, 1...k-matrix A can have and still avoid P, where P is a 0, 1...k-matrix.

Example

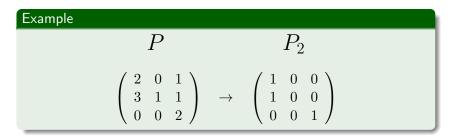
$$P = \left(\begin{array}{cc} 2 & 1\\ 1 & 1 \end{array}\right)$$

 $\mathrm{ex}_2(P,3)=14,$ because any 3×3 matrix with all entries 0,1,2 and sum over 14 contains P.

Mapping 0, 1...k-matrices to 0, 1-matrices

Definition

 P_j : The 0,1-matrix formed by mapping all entries with values $\geq j$ to 1 and entries $\leq j - 1$ to 0.



0, j-matrices

Lemma

Let P be a matrix with only 0, j entries. For any $k \ge j$, $ex_k(P, n) = (j - 1)n^2 + (k - j + 1)ex(P_j, n).$

0, j-matrices

Lemma

Let P be a matrix with only 0, j entries. For any $k \ge j$, $ex_k(P, n) = (j - 1)n^2 + (k - j + 1)ex(P_j, n).$

Reasoning.

The 'optimal' matrix that avoids P has j-1 entries where an 'optimal' matrix that avoids P_j has 0's, and k entries where it has 1's. For example, an 'optimal' matrix that avoids some pattern P might look like:

$$\begin{pmatrix} j-1 & k & k \\ k & k & j-1 \\ k & j-1 & j-1 \end{pmatrix}$$

Calculating, this has $(j-1)n^2 + (k-j+1)ex(P_j,n)$ sum.

Simple Inequality

Theorem

$$(j-1)n^2 + (k-j+1)ex(P_j, n) \le ex_k(P, n) \le (j-1)n^2 + (k-j+1)ex(P_1, n)$$

where j is the maximum element in P .

Simple Inequality

Theorem

$$\begin{split} (j-1)n^2+(k-j+1)\mathrm{ex}(P_j,n) &\leq \mathrm{ex}_k(P,n) \leq \\ (j-1)n^2+(k-j+1)\mathrm{ex}(P_1,n) \\ \text{where } j \text{ is the maximum element in } P. \end{split}$$

Proof of LHS.

We find a matrix contained by P that has extremal function $(j-1)n^2 + (k-j+1)\exp(\frac{P_j}{p}, n)$.

Let P' be the P with all non-j entries replaced with 0's.

$$\begin{pmatrix} j & 0 & 1 \\ j & j-1 & 1 \\ 0 & 0 & j \end{pmatrix} \to \begin{pmatrix} j & 0 & 0 \\ j & 0 & 0 \\ 0 & 0 & j \end{pmatrix}$$

Simple Inequality

Theorem

$$\begin{split} (j-1)n^2+(k-j+1)\mathrm{ex}(P_j,n) &\leq \mathrm{ex}_k(P,n) \leq \\ (j-1)n^2+(k-j+1)\mathrm{ex}(P_1,n) \\ \text{where } j \text{ is the maximum element in } P. \end{split}$$

Proof of RHS.

We find a matrix that contains P that has extremal function $(j-1)n^2+(k-j+1)\mathrm{ex}(\underline{P_1},n).$

Let P' be the P with all non-0 entries replaced with 1's.

$$\begin{pmatrix} j & 0 & 1 \\ j & j-1 & 1 \\ 0 & 0 & j \end{pmatrix} \to \begin{pmatrix} j & 0 & j \\ j & j & j \\ 0 & 0 & j \end{pmatrix}$$

Some 0,1,2-matrices

$$n^2 + \exp(P_2, n) \le \exp_2(P, n) \le n^2 + \exp(P_1, n)$$

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Example Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\exp_2(P, n) = n^2 + 2n - 1 \leftarrow$ analogous to lower bound

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 $\exp(P,n) \ge n^2 + 3n - 2 \leftarrow \mathsf{NOT}$ analogous to lower bound

We generally believe that the lower bound is closer than the upper bound though.

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A general form

Theorem

A general form

Theorem

 $ex_2(P,n) = n^2 + O(n).$

Again serves as evidence that lower bound is better:

$$n^2 + \exp(P_2, n) \le \exp(P, n) \le n^2 + \exp(P_1, n)$$

$$n^2 + O(n) \le \exp(P, n) \le n^2 + O(n^{2-\epsilon})$$

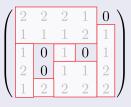
Theorem

Let P be a 0, 1, 2-matrix. The sum of numbers in a $n \times n$ matrix avoiding P is at most $\leq n^2 + O(kex(P_2, \frac{n}{\sqrt{k}}))$, where k is the number of 0's in the $n \times n$ matrix. Can be modified to $n^2 + O(\sqrt{kex(P_2, n)})$, easier to use but a bit weaker.

Improving the Simple Inequality

Proof Sketch.

Consider an $n \times n \ 0, 1, 2$ -matrix that $avoids \ P$. Build 'boxes' around the 0's as such:



We can limit the number and side length of the boxes to get the desired result.

Implications

Theorem

Let P be a 0, 1, 2-matrix. The sum of numbers in a $n \times n$ matrix avoiding P is at most $\leq n^2 + O(kex(P_2, \frac{n}{\sqrt{k}}))$, where k is the number of 0's in the $n \times n$ matrix. Can be modified to $n^2 + O(\sqrt{kex}(P_2, n))$, easier to use but a bit weaker.

- Bounds $ex_2(P, n)$ in terms of number of 0's in the $n \times n$ matrix that must avoid P: $ex_2(P, n) \le n^2 O(\sqrt{k}ex(P_2, n))$
- When k is (nontrvially) small, $O(\sqrt{k} \mathrm{ex}(P_2,n))$ is better than $\mathrm{ex}(P_1,n)$
- When k is large, it is not good enough.
- To make the result useful, we need to find another way to deal with the lots of k's case.

Characterize all the extremal functions $ex_k(P, n)$ in terms of 0,1 extremal functions ex(P, n).

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• For 0,1,2-matrices, find $\exp(P, n) - n^2$ upto a constant. We believe it is $\theta(\exp((P_2, n))$ rather than $\theta(P_1, n)$

Characterize all the extremal functions $ex_k(P, n)$ in terms of 0,1 extremal functions ex(P, n).

- For 0,1,2-matrices, find $\exp(P,n) n^2$ upto a constant. We believe it is $\theta(\exp((P_2, n))$ rather than $\theta(P_1, n)$
- Find the exact value of $ex_2(P, n)$ where

Let
$$P = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

Characterize all the extremal functions $ex_k(P, n)$ in terms of 0,1 extremal functions ex(P, n).

• Generalize the theorem about a row of 2's followed by rows of 1's to 0, 1...k-matrices, where we consider a row of *i*'s followed by rows of *j*'s where i > j.

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- Generalize the theorem about a row of 2's followed by rows of 1's to 0, 1...k-matrices, where we consider a row of *i*'s followed by rows of *j*'s where i > j.
- Generalize the last theorem (improvement on upper bound from simple inequality) to 0, 1...*k*-matrices.

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- My mentor Jesse Geneson
- The PRIMES Program
- My family